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Inseparable Galois Cohomology

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INTRODUCTION

Let C over A be a purely inseparable Galois ring extension of exponent one. Let R over A be a faithfully flat ring extension. In this paper, we give a simple explicit description for the (modified) Brauer group of central separable A-algebras split by both C and R. In case C over A is a simple field extension, this is given in [16, Theorems 3.5 and 3.6]. From this special case of our theorem, the complete theory of p-algebras due to Albert and others can be deduced.

Classically, if K is a Galois extension of F, the Brauer group $\mathscr{B}(K/F)$ is isomorphic to the group $E(K, G)$ of classes of group extensions of K^* by the Galois group G of K over F and to the second Galois cohomology group $H^2(G, K^*)$.

Serre asked whether one could obtain an analogous result when K over F is a purely inseparable exponent one field extension. Hochschild answered this in the affirmative by showing that the group extensions could be replaced by restricted Lie algebra extensions of K by the algebra of derivations D of K over F; he obtained an isomorphism between $\mathscr{B}(K/F)$ and $E(K, D)$, [10, p. 477].

Amitsur [2, p. 74] introduced a complex that enabled him to give a cohomological description of $\mathscr{B}(K/F)$ when K over F is a purely inseparable exponent one field extension. He proved that $\mathscr{B}(K/F)$ is isomorphic to $H^2(K/F, U)$, the second cohomology group of his complex.

In the exponent one purely inseparable case, Hochschild's results allow an additive description for $\mathscr{B}(K/F)$, namely, that $\mathscr{B}(K/F)$ is the quotient group of F modulo the image Z of a certain norm homomorphism $K^+ \xrightarrow{\nu} F^+$. Thus, for purely inseparable exponent one field extensions, the following isomorphisms were known:

$$\mathscr{B}(K/F) \cong H^2(K/F) \cong E(K, D) \cong F/Z.$$

The group F/Z has the special advantage that it is explicitly computable.

Recently, Yuan [18, pp. 427–450] generalized certain of these results to rings. For C, a purely inseparable exponent one Galois ring extension of A, he showed that $\mathcal{U}(C, A) \cong H^2(C/A) \cong E(C, D)$, where $\mathcal{U}(C, A)$ is the Chase–Rosenberg group [7, pp. 38–45] that coincides with the Brauer group when working over fields.

Of course, the natural question to ask was whether, in the purely inseparable ring case, a similar additive description is possible for $\mathcal{U}(C, A)$, $H^2(C/A)$, and $E(C, D)$. In the present paper, we show that this is indeed the case, thus obtaining the following generalization of the classical theory:

$$\mathcal{U}(C, A) \cong H^2(C/A) \cong E(C, D) \cong A/Z.$$

The importance of A/Z can be seen by observing the role that Z plays in the logarithmic derivative complex.

$$\mathcal{L}: (1) \to A^* \to C^* \xrightarrow{\delta_0} C^+ \xrightarrow{\gamma_0} A^+ \to (0),$$

where Z is defined as the image of γ_0. Thus, A/Z is a measure of the lack of exactness of the logarithmic derivative complex $\mathcal{L}$ at A^+. Since Yuan [17, p. 45] described the lack of exactness of $\mathcal{L}$ at C^+, our description of A/Z will complete the study of exactness of $\mathcal{L}$.

In the second part of the paper, we follow a homological approach to extend our earlier results. This is achieved by splicing together the two long exact sequences of cohomology of a certain four-term logarithmic complex to obtain the following exact sequence:

$$\begin{array}{ccc} (0) \to L(C/A) \to H^1(R/A, U) \to H^1(R \otimes C/C, U) \\ \downarrow \\ H^2(R \otimes C/C, U) \leftarrow H^2(R/A, U) \leftarrow J, \end{array}$$

where $L(C/A)$ denotes the cohomology at C^* of the logarithmic derivative complex $\mathcal{L}$ and J is an explicitly defined subgroup of A/Z. The image of J in $H^2(R/A, U)$, therefore, consists of all 2-cocycles split by both R and C. When R is a separable A-algebra, this result provides a very accurate picture of the interplay between separable and inseparable splitting rings of central separable algebras. Also, when R coincides with C, we recover the facts that $L(C/A) \cong H^1(C/A, U)$ and $A/Z \cong H^2(C/A, U)$.

We now proceed to define the modified Brauer group, $\mathcal{U}(C, A)$. Consider the collection of central separable A-algebras S that contain C as a splitting A-subalgebra; that is, S contains C as a maximal commutative A-subalgebra and S is projective as a C-module. Two such algebras S and T are called admissibly isomorphic, denoted by $S \simeq T$, if there exists an A-algebra isomorphism between S and T that pointwise fixes the elements of C. The relation "$\simeq$" is an equivalence relation on the set of central separable A-algebras containing C as a splitting A-subalgebra and we denote the equivalence class determined by such an S by $\langle S \rangle$. The set of all these admissible isomorphism classes is denoted by $\mathcal{O}\!\ell(C, A)$.

Given S and T central separable A-algebras containing C as a splitting A-subalgebra, we regard S and T as left C-modules, and we regard $S \otimes_C T$ as a right $S \otimes_A T$-module; that is, $S = {}_C S$, $T = {}_C T$, and $S \otimes_C T = (S \otimes {}_C T)_{S \otimes_A T}$. Then define

$$S * T = \operatorname*{End}_{S \otimes_A T}(S \otimes_C T).$$

$S * T$ turns out to be a central separable A-algebra containing C as a splitting A-subalgebra [7, p. 38]. This gives rise to a binary operation on $\mathcal{O}\!\ell(C, A)$ defined by

$$\langle S \rangle * \langle T \rangle = \langle S * T \rangle = \langle \mathrm{End}_{S \otimes_A T}(S \otimes_C T) \rangle.$$

In fact, this operation makes $\mathcal{O}\!\ell(C, A)$ into an abelian group whose identity is $\langle \mathrm{End}_A(C) \rangle$.

Now, this group $\mathcal{O}\!\ell(C, A)$ is obviously very closely related to the "classical" Brauer group $\mathscr{B}(C/A)$. In fact, by [3, p. 383], given that $[S] \in \mathscr{B}(C/A)$, there exists $\langle T \rangle \in \mathcal{O}\!\ell(C, A)$ such that $[S] = [T]$. The complete picture is given by the following exact sequence:

$$(0) \to \mathscr{P}(C/A) \to \mathscr{P}(A) \to \mathscr{P}(C) \xrightarrow{\sigma} \mathcal{O}\!\ell(C, A) \xrightarrow{\tau} \mathscr{B}(C/A) \to (0),$$

where $\mathscr{P}(A)$ and $\mathscr{P}(C)$ are the projective class groups of A and C, respectively, $\mathscr{P}(C/A)$ is the kernel of the canonical homomorphism from $\mathscr{P}(A)$ to $\mathscr{P}(C)$, the map σ is defined by

$$\sigma(M) = \langle \mathrm{End}_A(M) \rangle, \qquad \text{for } M \in \mathscr{P}(C)$$

and the map τ is defined by

$$\tau(\langle S \rangle) = [S], \qquad \text{for} \quad \langle S \rangle \in \mathcal{O}\!\ell(C, A).$$

From this exact sequence, we can easily see that when $\mathscr{P}(C) = (0)$, as when we work over fields, $\mathcal{O}\!\ell(C, A)$ coincides with $\mathscr{B}(C/A)$. Therefore, the results we obtain using $\mathcal{O}\!\ell(C, A)$ for rings will generalize the results using the Brauer group for fields.

The final portion of the paper is devoted to a computation of the Brauer group of a polynomial ring, generalizing a theorem of Auslander and Goldman [3, p. 389].

Throughout this paper, all rings are commutative and of characteristic p, unless specified otherwise, and the symbol $\otimes$ will mean $\otimes_A$, the tensor product over A.

1. The Determination of the Brauer Group

Let ∂ be a derivation on a (commutative) ring C of prime characteristic p and let A denote the kernel of ∂. Assume that C is a finitely generated projective A-module and that $\operatorname{End}_A(C) = C[\partial]$. Then, ∂ satisfies a polynomial

$$X(t) = \alpha_0 t + \alpha_1 t^p + \cdots + \alpha_i t^{p^i} + \cdots + \alpha_m t^{p^m},$$

with $\alpha_i \in A$. Moreover,

$$\{f \in C[t] \mid f(\partial) = 0\} = X(t) \cdot C[t] \qquad [17, \text{p. } 44].$$

Let $C[t; \partial]$ denote the differential polynomial ring defined by

$$tc - ct = \partial c, \qquad c \in C.$$

Now, by an inductive argument, we see that

$$t^r c = ct^r + \binom{r}{1} (\partial c)\, t^{r-1} + \binom{r}{2} (\partial^2 c)\, t^{r-2} + \cdots + (\partial^r c).$$

Thus, $t^p c = ct^p + \partial^p c$, so $X(t) - a$ lies in the center of $C[t; \partial]$, for each $a \in A$. Therefore, for each $a \in A$, we write

$$S = C[t; \partial]/(X(t) - a),$$

where $(X(t) - a)$ denotes the ideal of $C[t; \partial]$ generated by $X(t) - a$.

Theorem 1. *Let C be an A-algebra such that C is finitely generated and projective as an A-module and $\operatorname{End}_A(C) = C[\partial]$. Then, for any $a \in A$, the A-algebra $S = C[t; \partial]/(X(t) - a)$ is central separable and contains C as a splitting A-subalgebra.*

Proof. For each maximal ideal M of A, let $(\bar{X}(t) - \bar{a})$ denote the ideal of $C/MC[t; \bar{\partial}]$ generated by $\bar{X}(t) - \bar{a}$, where $\bar{\partial}$ is the canonical derivation on C/MC induced by ∂ and $\bar{X}(t) - \bar{a}$ denotes the image of $X(t) - a$ in $C/MC[t; \bar{\partial}]$.

Now, it is straightforward to verify that the mapping λ given by $\lambda(\overline{\sum_i c_i t^i}) = \overline{\sum_i (c_i + MC)\, t^i}$ is an isomorphism between S/MS and $C/MC[t; \bar{\partial}]/(\bar{X}(t) - \bar{a})$. This enables us, then, to restrict our attention to $C/MC[t; \bar{\partial}]/(\bar{X}(t) - \bar{a})$. The next step is to determine an important property of C/MC. The following lemma, given in more generality than is actually needed here, will help us to accomplish this.

Lemma 1. *Let R be a finite-dimensional algebra over a field F and let $\operatorname{End}_F(R) = R[\partial]$, where ∂ is an F-derivation on R. Then, no proper nontrivial ideal of R is stable under ∂.*

Proof. Suppose that J is a proper nontrivial ideal of R such that $\partial(J) \subseteq J$. Let $\{e_1, e_2, \ldots, e_l\}$ be an F-basis for J. Extend it to R so that $J = \sum_{i=1}^{l} \oplus Fe_i$ and $R = \sum_{i=1}^{n} \oplus Fe_i$, where $1 \leqslant l < n$. Define $f \in \mathrm{End}_F(R)$ such that $f(e_i) = e_{i+1}$, $i = 1, 2, \ldots, n-1$ and $f(e_n) = e_1$. Clearly, $f(J) \not\subseteq J$. But since $\mathrm{End}_F(R) = R[\partial]$, we have $f = \sum_{i=0}^{t} r_i \partial^i$. So for each $x \in J, f(x) = \sum_{i=0}^{t} r_i \partial^i(x)$. But each $r_i \partial^i(x)$ is an element of J for $i = 0, 1, 2, \ldots, t$. Hence, $f(J) \subseteq J$. This contradiction establishes our result.

Proof of Theorem (*continued*). To see that S is a central separable A-algebra, we begin by taking any $\alpha \in A/M$ and observing that $\alpha t = t\alpha$, so A/M is a subset of the center $\mathscr{H}$ of $C/MC[t; \bar{\partial}]/(\bar{X}(t) - \bar{a})$. Conversely, take any $d = \sum_{i=0}^{n} d_i t^i$, $n < p^m$, in $\mathscr{H}$. Then, for each $c \in C/MC$, the Lie product $[d, c] = 0$. However, $[d, c] = \sum_{i=1}^{n} d_i [\sum_{j=0}^{i} \binom{i}{j} \bar{\partial}^j c \cdot t^{i-j} - c \cdot t^i]$, showing that $\sum_{i=1}^{n} d_i \bar{\partial}^i c$, the coefficient of t^0, is zero for all $c \in C/MC$. Then, $\bar{\partial}$ satisfies a polynomial of degree $n < p^m$. Thus, $\mathscr{H}$ cannot have any element of positive degree, and $d = d_0 \in C/MC$. Moreover, this shows that $[d, t] = 0$, which gives $d \in A/M$. Therefore, $\mathscr{H} = A/M$.

Now, let I be a proper ideal of $C/MC[t; \bar{\partial}]/(\bar{X}(t) - \bar{a})$. Then $I = K/(\bar{X}(t) - \bar{a})$, where K is a proper ideal of $C/MC[t; \bar{\partial}]$. But $K \cap C/MC$ is a proper ideal of C/MC and it is straightforward to see that it is closed under $\bar{\partial}$. Thus, by Lemma 1, K has no nonzero elements in common with C/MC. Now, if $K \neq (\bar{X}(t) - \bar{a})$, the fact that the degree of $[m, c]$ is strictly less than the degree of m, for any $m \in K/(\bar{X}(t) - \bar{a})$, implies that $[m, c] = 0$ for each $c \in C/MC$, a contradiction.

The fact that S is projective as a C-module is obvious from its very definition. To see that S contains C as a splitting A-subalgebra, consider the canonical A-algebra, C-module homomorphism

$$\mu: C[t; \partial] \to C[\partial] = \mathrm{End}_A(C)$$

determined by $\mu(t) = \partial$. Now, for any $d = \sum_{i=0}^{n} d_i t^i$, where $n < p^m$ and some $d_i \neq 0$ for $i \geqslant 1$, we have $\mu(d)$ a nonzero element outside of C. But since $\mathrm{End}_A(C)$ is a central separable A-algebra containing C as a splitting A-subalgebra, there must exist some $c \in C$ such that $[c, \mu(d)] \neq 0$. But this immediately implies that $[c, d] \neq 0$. Thus, S contains C as a splitting A-subalgebra and our proof is complete.

We now define a mapping $\Psi: A \to \mathscr{Cl}(C, A)$ by $\Psi(a) = \langle C[t; \partial]/(X(t) - a) \rangle$. Theorem 1 actually says that our mapping Ψ exists. The following theorem completely describes the form of central separable A-algebras with C as a splitting A-subalgebra and will enable us to conclude that Ψ is surjective.

Theorem 2. *Let C be an A-algebra such that C is finitely generated and projective as an A-module and* $\mathrm{End}_A(C) = C[\partial]$. *Let R be a central separable*

A-algebra with C as a splitting A-subalgebra. Then, R is of the form $C[t; \partial]/(X(t) - a)$, for some $a \in A$.

Proof. Let R be a central separable A-algebra with C as a splitting A-subalgebra. By [8], ∂ can be extended to an inner derivation on R, determined by some $\rho \in R$. It is straightforward to show that $X(\rho) \cdot c = c \cdot X(\rho)$ for all $c \in C$, so $X(\rho) \in C$. Then, by applying ∂, we obtain $\partial(X(\rho)) = 0$, so $X(\rho) \in A$. Thus, $\Psi(X(\rho)) = \langle C[t; \partial]/(X(t) - X(\rho))\rangle$. Now, if we let T denote $C[t; \partial]/(X(t) - X(\rho))$, then the A-subalgebra of R generated by C and ρ is a canonical homomorphic image of T. It follows that $T \cong C[\rho]$ and that $C[\rho]$ has the same A-rank as R. Therefore, $T \cong R$ and we obtain the desired result.

THEOREM 3. *The map $\Psi: A \to \mathcal{Cl}(C, A)$, defined by*

$$\Psi(a) = \langle C[t; \partial]/(X(t) - a)\rangle,$$

is a group epimorphism.

Proof. Let S^{a_i} denote $C[t; \partial]/(X(t) - a_i)$, for $i = 1, 2$, and let S denote $C[t; \partial]/(X(t) - (a_1 + a_2))$. If $U = S^{a_1} \otimes S^{a_2}$ and if $V = S^{a_1} \otimes_C S^{a_2}$, then $\mathrm{End}_{U^0} V$ is the product $S^{a_1} * S^{a_2}$ used in $\mathcal{Cl}(C, A)$. We denote $\mathrm{End}_{U^0} V$ by B, so $B = S^{a_1} * S^{a_2}$. Our aim is to show that B is isomorphic to S, for this will establish our theorem.

Using the description of B as given in [18, p. 436], we see that the A-map defined by $t \xrightarrow{f} t \otimes 1 + 1 \otimes t$ takes S to B. But the kernel of this map is of the form βS, where β is an ideal of A, [3, p. 375]. But for any $b \in \beta$, we have $0 = f(b \cdot 1) = b$, so the kernel of f is (0). This means that S can be identified with a subring of B.

Since we now have a two-sided S-module structure on B, we obtain $B \cong \mathrm{Hom}_{S^e}(S, B) \otimes_A S = B^S \otimes_A S$. However, $B^S \subset C$ and each element of B^S commutes with each element of S, so B^S is contained in the center of S, which is A. But this directly implies that $B^S = A$, so we obtain $B \cong S$.

Finally, using this result, we can directly compute $\Psi(a_1 + a_2)$ to obtain $\Psi(a_1 + a_2) = \langle S\rangle = \langle B\rangle = \Psi(a_1) * \Psi(a_2)$, which completes the proof.

In [17, p. 41] Yuan describes the logarithmic derivative complex

$$\mathscr{L}: (1) \to A^* \xrightarrow{j} C^* \xrightarrow{\delta_0} C^+ \xrightarrow{\gamma_0} A^+ \to (0),$$

where A^* and C^* denote the groups of units of A and C, respectively, A^+ and C^+ denote the additive groups of the respective rings, j is the canonical inclusion, δ_0 is the logarithmic derivative homomorphism defined by

$$\delta_0(c) = \partial c/c, \qquad c \in C^*,$$

and γ_0 is the additive homomorphism from C^+ to A^+ defined by

$$\gamma_0(c) = \sum_{i=0}^{n} a_i([\partial^{p^i-1}c] + [\partial^{p^{i-1}-1}c]^p + \cdots + [\partial^{p^{i-j}-1}c]^{p^j} + \cdots + c^{p^i}), \quad c \in C.$$

The map γ_0 has the property that $X(\partial + \Lambda c) = \Lambda\gamma_0(c)$ for each $c \in C$, where Λ denotes the (left) multiplication map, [17, p. 41] and [5, p. 201]. The homology group of $\mathscr{L}$ at C^+ is $\operatorname{Ker}\gamma_0/\operatorname{Im}\delta_0$ and is referred to as the logarithmic derivative group, $L[C/A)$. In [17, p. 45], Yuan found $L(C/A)$ to be isomorphic to the projective class group $\mathscr{P}(C/A)$, thus providing a measure of the lack of exactness of $\mathscr{L}$ at C^+ in terms of the classes of rank one projective A-modules split by C. This result, of course, extends the corresponding result of Jacobson [11, p. 224] because $\operatorname{Ker}\gamma_0 = \operatorname{Im}\delta_0$ when C is a finite-dimensional field extension over A and $X(t)$ is the characteristic polynomial for ∂.

One of our aims in this paper is to measure how far we are from exactness at A in the logarithmic derivative complex $\mathscr{L}$. We denote the image of γ_0 by Z and we compute A/Z.

THEOREM 4. *The sequence of groups* $C \xrightarrow{\gamma_0} A \xrightarrow{\Psi} \mathcal{O}\!\ell(C, A) \to (0)$ *is exact.*

Proof. Take any $z \in Z$. Then, there exists $c \in C$ such that $\gamma_0(c) = z$. It is straightforward to show that the mapping $\lambda\colon C[t; \partial]/(X(t) - z) \to C[t; \partial]/(X(t))$ determined by $\lambda(t) = t + c$ is an A-algebra isomorphism, so $Z \subseteq \operatorname{Ker}\Psi$.

Now, for any $a \in \operatorname{Ker}\Psi$, there is an A-algebra isomorphism

$$\mathscr{F}\colon C[t; \partial]/(X(t) - a) \to \operatorname{End}_A(C)$$

with the property that for each $c \in C$, $\mathscr{F}(c) = \Lambda c$, where Λc denotes left multiplication by c. It is easy to see that $(\mathscr{F}(t) - \partial)\,\Lambda c = \Lambda c(\mathscr{F}(t) - \partial)$, and since $\mathscr{F}(C)$ is a splitting A-subalgebra of $\operatorname{End}_A(C)$, we obtain $\mathscr{F}(t) - \partial = \Lambda z$, for some $z \in C$. Then $\gamma_0(z) = a$, and our proof is complete.

COROLLARY. *Let C be an A-algebra such that C is finitely generated and projective as an A-module and* $\operatorname{End}_A(C) = C[\partial]$; *let Z denote the image of the map $\gamma_0\colon C \to A$ of the logarithmic derivative complex; let $\mathcal{O}\!\ell(C, A)$ denote the Chase–Rosenberg "modified Brauer group"; let $H^2(C/A, U)$ denote the second Amitsur cohomology group of C over A; and let $E(C, D)$ denote Hochschild's group of regular restricted Lie algebra extensions of C by D. Then*

$$A/Z \cong \mathcal{O}\!\ell(C, A) \cong H^2(C/A, U) \cong E(C, D).$$

Remark 1. In the special case when C over A is a simple field extension, the isomorphism

$$\mathcal{O}\!\ell(C, A) \cong A/\{\partial^{p-1}c + c^p \mid c \in C\}$$

is due to Hochschild [10, p. 489]. A different proof for this special case may be found in the book by Demazure and Gabriel on algebraic groups [8, p. 470, Corollaire 9.9].

Remark 2. The theory developed here can also be formulated in terms of the Cartier operator that is closely related to our map γ_0. The approach given here has the advantage of being very explicit.

2. Splitting Rings of Central Separable Algebras

A basic theorem in the classical theory of simple algebras is that every central simple algebra has a separable splitting field. In [1, p. 276], Amitsur gave a description for the Brauer group of central simple A-algebras split by a pair of Galois extensions C and R. If R is a separable field extension and C a simple purely inseparable field extension, an additive formula for this Brauer group is given in [16, Theorem 3.6] and this formula plays a key role in the theory developed there. In the following, we extend this formula to the setting of commutative rings.

Let F be a covariant functor from the category of (commutative) A-algebras to the category of abelian groups. Given an A-algebra R, the Amitsur complex $\mathscr{C}(R/A, F)$ of R/A with coefficients in F is defined as follows:

$$\mathscr{C}^n(R/A, F) = F(R^{n+1}), \qquad n \geqslant 0$$

and the coboundary operator

$$\Delta_n : F(R^{n+1}) \to F(R^{n+2})$$

is defined by

$$\Delta_n = \sum_{i=0}^{n+1} (-1)^i F(E_i),$$

where E_i is the ith face operator

$$E_i : R^{n+1} \to R^{n+2},$$

given by

$$x_0 \otimes \cdots \otimes x_n \longmapsto x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n .$$

The nth cohomology group of $\mathscr{C}(R/A, F)$ is denoted by $H^n(R/A, F)$. In the following, G will denote the functor that assigns to a given A-algebra R its underlying additive group R^+ and U will denote the functor that assigns to each such R its group R^* of invertible elements.

Now, as before, let ∂ be a derivation on a (commutative) ring C of prime characteristic p and denote by A the kernel of ∂. Assume that C is finitely generated and projective as an A-module and that $\mathrm{Hom}_A(C, C) = C[\partial]$. According to [16, p. 284], for any A-algebra R, there is a four-term complex

$$(0) \to R^* \to (R \otimes C)^* \to R \otimes C \to R \to (0).$$

Replacing R by $R \otimes R$, $R \otimes R \otimes R$, etc., it is straightforward to show that we obtain the sequence of complexes

$$(0) \to \mathscr{C}(R/A, U) \to \mathscr{C}(R \otimes C/C, U) \xrightarrow{\delta} \mathscr{C}(R \otimes C/C, G) \xrightarrow{\gamma} \mathscr{C}(R/A, G) \to (0).$$

Let $\mathscr{D}$ denote the image of δ and let $\mathscr{G}$ denote the cokernel of δ. Thus, we obtain two short exact sequences of complexes:

$$(0) \to \mathscr{C}(R/A, U) \to \mathscr{C}(R \otimes C/C, U) \to \mathscr{D} \to (0), \tag{1}$$

and

$$(0) \to \mathscr{D} \to \mathscr{C}(R \otimes C/C, G) \to \mathscr{G} \to (0). \tag{2}$$

By passing to cohomology, (1) and (2) yield the exact sequences

$$\begin{array}{c} (0) \to A^* \to C^* \xrightarrow{\delta} H^0(\mathscr{D}) \to H^1(R/A, U) \to H^1(R \otimes C/C, U) \\ \downarrow \\ H^3(R/A, U) \leftarrow H^2(\mathscr{D}) \leftarrow H^2(R \otimes C/C, U) \leftarrow H^2(R/A, U) \leftarrow H^1(\mathscr{D}), \end{array} \tag{3}$$

and

$$\begin{array}{c} (0) \to H^0(\mathscr{D}) \to H^0(R \otimes C/C, G) \to H^0(\mathscr{G}) \to H^1(\mathscr{D}) \\ \downarrow \\ H^2(R \otimes C/C, G) \leftarrow H^2(\mathscr{D}) \leftarrow H^1(\mathscr{G}) \leftarrow H^1(R \otimes C/C, G). \end{array} \tag{4}$$

THEOREM 5. *Assume that R is faithfully flat as an A-module and that the Picard groups $\mathscr{P}(R \otimes C/R)$ and $\mathscr{P}(R^2 \otimes C/R^2)$ are trivial. Then, the following sequence is exact:*

$$\begin{array}{c} (0) \to L(C/A) \to H^1(R/A, U) \to H^1(R \otimes C/C, U) \\ \downarrow \\ H^2(R \otimes C/C, U) \leftarrow H^2(R/A, U) \leftarrow A \cap \mathscr{G}^0/Z, \end{array}$$

where Z denotes the image of γ and $L(C/A)$ denotes the logarithmic derivative group.

We first record the special case $R = C$ as a

COROLLARY. *$H^1(C/A, U) \cong L(C/A)$ and $H^2(C/A, U) \cong A/Z$.*

Proof of Theorem. Since R is faithfully flat over A, $\mathscr{C}(R \otimes C/C, G)$ is cohomologically trivial [14, Lemma 2.2, p. 224]. Thus, from (4), we obtain the exactness of

$$(0) \to H^0(\mathscr{D}) \to C \to H^0(\mathscr{G}) \to H^1(\mathscr{D}) \to (0). \tag{5}$$

Now, it follows from the assumption on the Picard groups that the two lower rows of the diagram below are exact:

$$\begin{array}{ccccc}
(R^3 \otimes C)^* & \xrightarrow[\delta]{} & R^3 \otimes C & \to & R^3 \\
\uparrow & & \uparrow & & \uparrow \\
(R^2 \otimes C)^* & \xrightarrow[\delta]{} & R^2 \otimes C & \to & R^2 \\
\uparrow & & \uparrow & & \uparrow \\
(R \otimes C)^* & \xrightarrow[\delta]{} & R \otimes C & \to & R.
\end{array}$$

So $\mathscr{G}^i$, $i = 1, 2$, is just the image of $R^i \otimes C$ in R^i.

From the commutativity of the diagram

$$\begin{array}{ccc}
\mathscr{G}^2 & \longrightarrow & R^3 \\
\uparrow & & \uparrow \\
\mathscr{G}^1 & \xrightarrow[1-1]{} & R^2 \\
\uparrow & & \uparrow \\
\mathscr{G}^0 & \xrightarrow[1-1]{} & R,
\end{array}$$

we obtain $H^0(\mathscr{G}) = A \cap \mathscr{G}^0$, because $H^0(R/A, G)$ is just A. In (5), if we replace $H^0(\mathscr{G})$ by $A \cap \mathscr{G}^0$, we obtain the exactness of

$$(0) \to H^0(\mathscr{D}) \to C \xrightarrow{\delta} (A \cap \mathscr{G}^0) \to H^1(\mathscr{D}) \to (0).$$

In particular,

$$H^0(\mathscr{D}) = \operatorname{Kernel}\{C \xrightarrow{\delta} A\},$$

and

$$H^1(\mathscr{D}) = (A \cap \mathscr{G}^0)/Z, \qquad \text{where} \qquad Z = \operatorname{Image}\{C \xrightarrow{\gamma} A\}.$$

By substituting these two equalities into the exact sequence (3), we have the exactness of the sequence

$$\begin{array}{l}
(0) \to A^* \to C^* \xrightarrow[\delta]{} (\operatorname{Kernel} \gamma) \to H^1(R/A, U) \to H^1(R \otimes C/C, U) \\
\hfill \downarrow \\
\qquad\qquad H^2(R \otimes C/C, U) \leftarrow H^2(R/A, U) \leftarrow (A \cap \mathscr{G}^0)/Z,
\end{array}$$

which of course completes the proof of the theorem.

In the special case when $R = C$, $H^i(C \otimes C/C, U)$ is trivial for each $i > 0$, since the kernel of the mapping $C \otimes C \to C$, given by $x \otimes y \to xy$, is nilpotent [14, Proposition 3.3, p. 225]. It follows from this that $H^1(C/A, U)$ is isomorphic to $L(C/A)$ and $H^2(C/A, U)$ is isomorphic to $A \cap \mathscr{G}^0/Z$. We show that $\mathscr{G}^0$ coincides with C, thus obtaining the result that $H^2(C/A, U)$ is isomorphic to A/Z. Now, according to Theorem 4 of the preceding section, the cokernel of the map $C^2 \xrightarrow{\gamma} C$ is isomorphic to $\mathscr{Cl}(C \otimes C, C)$. Again, since $C \otimes C$ is obtained from C by adjoining nilpotent elements, $\mathscr{Cl}(C \otimes C, C)$ is trivial [19, Lemma 3]. This shows that γ is an epimorphism, as desired.

3. The Brauer Group of a Polynomial Ring

Given a polynomial ring $A = K[t_1, \text{——}, t_r]$ over a field K, we have two K-algebra homomorphisms

$$K \hookrightarrow A \xrightarrow{\varphi} K, \qquad \text{where } \varphi \text{ is given by } t_i \mapsto 0.$$

The induced map $\varphi_*: \mathscr{B}(A) \to \mathscr{B}(K)$ is onto because the restriction of φ to K is the identity map. According to [3, p. 391], φ_* is an isomorphism on all l-primary components with l distinct from the characteristic p of K.

THEOREM 6. *The map* $\varphi_*: \mathscr{B}(A)_p \to \mathscr{B}(K)$ *is an isomorphism if and only if* $r = 1$ *and* K *is perfect.*

Proof. If $r > 1$, we claim that φ_* is never one–one. Let B denote the A-algebra $A[s]$ defined by the sole relation $s^p = t_1$. Since A is a unique factorization domain, the Brauer group $\mathscr{B}(B/A)$ is isomorphic to the modified one that is just $A/\{\partial^{p-1}x + x^p \mid x \in B\}$, where ∂ is the A-derivation on B defined by $\partial s = 1$. Now, it is easy to verify that t_r is not zero modulo $\{\partial^{p-1}x + x^p \mid x \in B\}$. Since φ_* takes the central division algebra defined by t_r to zero, it is not one–one.

Now, assume that $r = 1$ and that K is not perfect. So K has a simple purely inseparable field extension $E = K[\sigma]$ with $\sigma^p \in K$. If ∂ is the $A = K[t_1]$-derivation on $E[t_1]$ defined by $\partial\sigma = 1$, exactly the same computation as above shows the nontrivial algebra class given by t_1 is mapped to zero by φ_*. Again, φ_* is not one–one. Thus, for φ_* to be one–one, the conditions $r = 1$ and K being perfect are necessary. That these two conditions are sufficient is, of course, proved in [3, p. 389].

Remark. The referee of this paper has informed me that the preceding result is also known to Goldman (unpublished).

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